6.1
Transformation Formulas

Learning objectives:

1. To derive trigonometric product formulas
2. To derive trigonometric sum and difference formulas
   And
3. To practice the related formulas.

Product Formulas

We have already derived the relations

$$\sin(\alpha+\beta) = sin\alpha\, cos\beta + cos\alpha\, sin\beta$$

$$\sin(\alpha-\beta) = sin\alpha\, cos\beta - cos\alpha\, sin\beta$$

Adding and subtracting, we get

$$2\, sin\alpha\, cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad \ldots (i)$$

$$2\, cos\alpha\, sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta) \quad \ldots (ii)$$

Similarly, from the relations

$$\cos(\alpha+\beta) = cos\alpha\, cos\beta - sin\alpha\, sin\beta \quad \ldots (iii)$$

$$\cos(\alpha-\beta) = cos\alpha\, cos\beta + sin\alpha\, sin\beta \quad \ldots (iv)$$

we get by adding and subtracting

$$2\, cos\alpha\, cos\beta = \cos(\alpha+\beta) + cos\,(\alpha-\beta)$$

$$-2\, sin\alpha\, sin\beta = \cos(\alpha+\beta) - cos\,(\alpha-\beta)$$

These formulas transform products of sines and cosines into sums or differences of sines or cosines.

Sum and Difference Formulas

The sum and difference formulas are obtained from the product formulas by setting

$$\alpha+\beta = A$$

$$\alpha-\beta = B$$

Adding and subtracting, we get

$$\alpha = \frac{A+B}{2}$$

$$\beta = \frac{A-B}{2}$$

Substituting these values into product formulas, we obtain

$$sinA + sinB = 2\,sin\frac{A+B}{2}\;cos\frac{A-B}{2} \qquad \ldots(v)$$

$$sinA - sinB = 2\,cos\frac{A+B}{2}\;sin\frac{A-B}{2} \qquad \ldots(vi)$$

$$cosA + cosB = 2\,cos\frac{A+B}{2}\,cos\frac{A-B}{2} \qquad \ldots(vii)$$

$$cosA - cosB = -2\,sin\frac{A+B}{2}\,sin\frac{A-B}{2} \qquad \ldots(viii)$$

These formulas transform sums or differences of sines and cosines into products of sines and cosines.

Example1:

Express each of the following as a sum or difference.

$a. sin40^\circ\, cos30^\circ$
$b. cos110^\circ\, sin55^\circ$

Solution:

$a.$

$$2sin40^\circ\, cos30^\circ = sin(40^\circ + 30^\circ) + sin(40^\circ - 30^\circ)$$
$$= sin70^\circ + cos10^\circ$$

$b.$

$$2cos110^\circ\, sin55^\circ = sin(110^\circ + 55^\circ) - sin(110^\circ - 55^\circ)$$
$$= sin165^\circ - sin55^\circ$$

Example 2:

Express each of the following as a product.

$a. sin50^\circ + sin40^\circ$
$b. sin70^\circ - sin20^\circ$

Solution:

a.

$$sin50^\circ + sin40^\circ = 2sin\tfrac{1}{2}(50^\circ + 40^\circ)\,cos\tfrac{1}{2}(50^\circ - 40^\circ)$$
$$= 2sin45^\circ\, cos5^\circ = \sqrt{2}\,cos5^\circ$$

b.

$$sin70^\circ - sin20^\circ = 2cos\tfrac{1}{2}(70^\circ + 20^\circ)\,sin\tfrac{1}{2}(70^\circ - 20^\circ)$$
$$= 2cos45^\circ\, sin25^\circ = \sqrt{2}\,sin25^\circ$$

Example 3:

Prove $\frac{\sin 4A+\sin 2A}{\cos 4A+\cos 2A} = \tan 3A$

Solution:

$$\frac{\sin 4A+\sin 2A}{\cos 4A+\cos 2A} = \frac{2\,sin\frac{4A+2A}{2}cos\frac{4A-2A}{2}}{2\,cos\frac{4A+2A}{2}cos\frac{4A-2A}{2}}$$
$$= \frac{2\sin 3A\; cosA}{2\cos 3A\cos A} = tan3A$$

P1.

Prove that $\sin 21^{\circ} \cos 9^{\circ} - \cos 84^{\circ} \cos 6^{\circ} = \frac{1}{4}$

Solution:

$$\sin 21^{\circ} \cos 9^{\circ} - \cos 84^{\circ} \cos 6^{\circ} =$$

$$= \frac{1}{2}\left(2 \sin 21^{\circ} \cos 9^{\circ} - 2\cos 84^{\circ} \cos 6^{\circ}\right)$$

$$= \frac{1}{2}\left(\sin(21^{\circ} + 9^{\circ}) + \sin(21^{\circ} - 9^{\circ}) - 2\cos(90^{\circ} - 6^{\circ}) \cos 6^{\circ}\right)$$

$$= \frac{1}{2}\left(\sin 30^{\circ} + \sin 12^{\circ} - \sin 12^{\circ}\right) = \frac{1}{4}$$

P2.

Prove that $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta} = 4\cos 2\theta \cos 4\theta$

Solution:

$$\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = \frac{\sin\ 5\theta \cos 3\theta + \sin 3\theta \cos 5\theta}{\sin\ 5\theta \cos 3\theta - \sin 3\theta \cos 5\theta}$$

$$= \frac{\sin 8\theta}{\sin 2\theta}$$

$$= \frac{2 \sin 4\theta \cos 4\theta}{\sin 2\theta}$$

$$= \frac{4 \sin 2\theta \cos 2\theta \cos 4\theta}{\sin 2\theta}$$

$$= 4 \cos 2\theta \cos 4\theta$$

P3)

Prove that $\frac{cos7x+cos5x}{sin7x-sin5x} = cotx.$

Solution:

$$\frac{cos7x + cos5x}{sin7x - sin5x} = \frac{cos7x + cos5x}{sin7x - sin5x}$$

$$= \frac{2cos\frac{7x+5x}{2}\,cos\frac{7x-5x}{2}}{2cos\frac{7x+5x}{2}\,sin\frac{-7x-5x}{2}} = \frac{2cos\ 6x\ cosx}{2cos\ 6x\ sinx}$$

$$= \frac{\cos x}{sinx} = cotx$$

P4.

Find the value of $\frac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A.$

Solution:

$$\frac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A}$$

$$= \frac{\sin A + \sin 9A - (\sin 5A + \sin 13A)}{\cos A + \cos 13A - (\cos 5A + \cos 9A)}$$

$$= \frac{2\sin 5A \cos 4A - (2\sin 9A \cos 4A)}{2\cos 7A \cos 6A - (2\cos 7A \cos 2A)}$$

$$= \frac{\cos 4A}{\cos 7A}\left(\frac{\sin 5A - \sin 9A}{\cos 6A - \cos 2A}\right)$$

$$= \frac{\cos 4A}{\cos 7A}\left(\frac{2\cos 7A \sin(-2A)}{2\sin 4A \sin(-2A)}\right)$$

$$= \cot 4A$$

IP1.

Prove that $cos^2 76^\circ + cos^2 16^\circ - cos76^\circ cos16^\circ = \frac{3}{4}$

Solution:

$cos^2 76^\circ + cos^2 16^\circ - cos76^\circ cos16^\circ =$

$= cos^2 76^\circ + 1 - sin^2 16^\circ - \frac{1}{2}\left(2cos76^\circ cos16^\circ\right)$

$= 1 + cos^2 76^\circ - sin^2 16^\circ - \frac{1}{2}(\cos(76^\circ + 16^\circ) + \cos\,(76^\circ - 16^\circ))$

$= 1 + \cos(76^\circ + 16^\circ)\cos(76^\circ - 16^\circ) - \frac{1}{2}(\cos 92^\circ + \cos\ 60^\circ)$

Since $cos^2 A - sin^2 B = \cos(A + B)\cos\,(A - B)$

$= 1 + \cos 92^\circ \cos 60^\circ - \frac{1}{2}(\cos 92^\circ + \cos\ 60^\circ)$

$= 1 + \frac{1}{2}\cos 92^\circ - \frac{1}{2}\cos 92^\circ - \frac{1}{4}$

$= 1 - \frac{1}{4} = \frac{3}{4}$

IP2.

Prove that

$$\frac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \tan 3\theta$$

Solution:

$$\frac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \frac{2\cos 5\theta + \cos 3\theta + \cos 7\theta}{2\cos 3\theta + \cos\theta + \cos 5\theta}$$

$$= \frac{2\cos 5\theta + 2\cos 5\theta \cos 2\theta}{2\cos 3\theta + 2\cos 3\theta \cos 2\theta}$$

$$= \frac{\cos 5\theta (1 + \cos 2\theta)}{\cos 3\theta (1 + \cos 2\theta)}$$

$$= \frac{\cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta}{\cos 3\theta}$$

$$= \cos 2\theta - \tan 3\theta \sin 2\theta$$

IP3)

Prove that $\frac{sin5x-2sin3x+sinx}{cos5x-cosx} = tanx.$

Solution:

$$\frac{sin5x - 2sin3x + sinx}{cos5x - cosx} = \frac{2\ sin3x\ cos2x - 2sin3x}{-2sin3x\ sin2x}$$

$$= \frac{1 - cos2x}{sin2x}$$

$$= \frac{2\ sin^2 x}{2\ sinx\ cosx} = tanx$$

IP4.

Show that $\frac{\cos 6\theta+6\cos 4\theta+15\,cos2\theta+10}{\cos 5\theta+5cos3\theta+10\,cos\theta} = 2\,cos\theta$

Solution:

$$\frac{\cos 6\theta + 6\cos 4\theta + 15\,cos2\theta + 10}{\cos 5\theta + 5cos3\theta + 10\,cos\theta}$$

$$= \frac{(\cos 6\theta + \cos 4\theta) + 5cos4\theta + 5\,cos2\theta + 10cos2\theta + 10}{\cos 5\theta + 5cos3\theta + 10\,cos\theta}$$

$$= \frac{2\cos5\theta\,cos\theta + 5(2cos3\theta cos\theta) + 10(cos2\theta + 1)}{\cos 5\theta + 5cos3\theta + 10\,cos\theta}$$

$$= \frac{2\cos5\theta\,cos\theta + 10cos3\theta cos\theta + 10(2cos^2\theta)}{\cos 5\theta + 5cos3\theta + 10\,cos\theta}$$

$$= \frac{2\,cos\theta(\cos5\theta + 5cos3\theta + 10cos\theta)}{\cos 5\theta + 5cos3\theta + 10\,cos\theta} = 2cos\theta$$

Exercises

I. Prove the following identities.

1) $\frac{sin7\theta - sin5\theta}{cos7\theta + cos5\theta} = tan\theta$

2) $\frac{cos\ 2B + \cos 2A}{cos\ 2B - \cos 2A} = \cot(A+B)\cot(A-B)$

3) $\frac{sin\ A + \sin 2A}{cosA - cos2A} = cot\frac{A}{2}$

4) $\cos(A+B) + sin(A-B)$

$$= 2\sin(45^{\circ} + A)\cos(45^{\circ} + B)$$

5) $\frac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A) +} = \tan(A+B)$

6) $\frac{sin\ A + 2\sin 3A + sin\ 5A}{\sin 3A + 2\ sin\ 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$

7) $\frac{sin\ A + \sin B}{sin\ A - \sin B} = \tan\frac{A+B}{2}\cot\frac{A-B}{2}$

8) $\frac{sin\ A + \sin B}{cos\ A - \cos B} = \tan\frac{A+B}{2}$

9) $\cos 3A + \cos 5A + \cos 7A + \cos 15A$

$$= 4\cos 4A \cos 5A \cos 6A$$

II. Express as a sum or difference of the following

1) $2\sin 5\theta \sin 7\theta$
2) $2\cos 7\theta \sin 5\theta$
3) $2\cos 11\theta \cos 3\theta$
4) $2\sin 54^{\circ} \sin 66^{\circ}$

6.2

Conditional Trigonometric Identities

Learning objectives:

1. To derive conditional trigonometric identities.
   And
2. To practice the related problems.

Conditional Identities

Trigonometric identities are equations involving trigonometric functions of angles, which are satisfied by all values of the angles for which the functions are defined.

If the angles involved in the identities are the three angles A, B and C of a triangle, then they are constrained by the relation $A+B+C=180^{\circ}$,then these trigonometric identities are known as conditional identities.

Some Identities

i) If $A+B+C=180^{\circ}$, then prove that

$$\sin 2A+\sin 2B+\sin 2C=4\ sinA\ sinB\ sinC$$

Proof:

$$\sin 2A+\sin 2B+\sin 2C$$

$$=2\sin(A+B)cos\,(A-B)+2sinC\ cosC$$

$$=2\sin(180^{\circ}-C)cos\,(A-B)$$
$$+2sinC\cos\,(180^{\circ}-(A+B))$$

$$=2\sin C\ cos\,(A-B)+2sinC\cos\,(A+B)$$
$$=2\sin C\,[cos(A-B)-\cos\,(A+B)]$$

$$=2sinC.\,2sinA\ \text{sinB}$$

$$=4sinA\ sinB\ \text{sinC}$$

ii) If $A+B+C=180^{\circ}$, then prove that

$$cosA+cosB-cosC=-1+4cos\tfrac{A}{2}\ cos\tfrac{B}{2}sin\tfrac{C}{2}\ .$$

Proof:

$$cosA+cosB-cosC$$

$$=cosA+(cosB-cosC)$$

$$=2cos^2\tfrac{A}{2}-1-2sin\tfrac{B+C}{2}\ sin\tfrac{B-C}{2}$$

$$=2cos^2\tfrac{A}{2}-1-2sin\left(90^{\circ}-\tfrac{A}{2}\right)sin\tfrac{B-C}{2}$$

$$=2cos\tfrac{A}{2}\left[cos\tfrac{A}{2}-sin\tfrac{B-C}{2}\right]-1$$

$$=2cos\tfrac{A}{2}\left[cos\tfrac{(180^{\circ}-(B+C))}{2}-sin\tfrac{B-C}{2}\right]-1$$

$$=2cos\tfrac{A}{2}\left[sin\tfrac{(B+C)}{2}-sin\tfrac{B-C}{2}\right]-1$$

$$=2cos\tfrac{A}{2}\ .2cos\tfrac{B}{2}sin\tfrac{C}{2}-1$$

$$=-1+4cos\tfrac{A}{2}\ cos\tfrac{B}{2}sin\tfrac{C}{2}$$

iii) If $\alpha + \beta + \gamma = \pi$, then prove that

$$sin^2\alpha + sin^2\beta - sin^2\gamma = 2\, sin\alpha\, sin\beta\, cos\gamma$$

Proof:

$$\alpha + \beta + \gamma = \pi \Rightarrow \alpha + \beta = \pi - \gamma$$

$$\Rightarrow \cos(\alpha + \beta) = \cos(\pi - \gamma) = -\cos\gamma$$

$$\Rightarrow \cos\alpha\, cos\beta - sin\alpha\, sin\beta = -\cos\gamma$$

$$\Rightarrow (\, sin\alpha\, sin\beta - cos\gamma)^2 = cos^2\alpha\, cos^2\beta$$

$$\Rightarrow sin^2\alpha\, sin^2\beta + cos^2\gamma - 2\, sin\alpha\, sin\beta\, cos\gamma$$

$$= (1 - sin^2\alpha)(1 - sin^2\beta)$$

$$\Rightarrow sin^2\alpha + sin^2\beta + cos^2\gamma - 1 = 2\, sin\alpha\, sin\beta\, cos\gamma$$

$$\Rightarrow sin^2\alpha + sin^2\beta - sin^2\gamma = 2\, sin\alpha\, sin\beta\, cos\gamma$$

iv) In a triangle ABC, then prove that

$$\tan A + tanB + tanC = tanA\, tanB\, tanC$$

Proof:

We have, $A + B + C = \pi \Rightarrow A + B = \pi - C$

$$\tan(A + B) = \tan(\pi - C) = -\tan C$$

$$\Rightarrow \tan A + tanB = -(1 - tanA\, tanB)\, tanC$$

$$\Rightarrow \tan A + tanB + tanC = tanA\, tanB\, tanC$$

v) If $A + B + C = \pi$, then prove that
$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$

Proof:

$$A + B + C = \pi \Rightarrow A + B = \pi - C$$

$$\Rightarrow \cot(A + B) = \cot\,(\pi - c)$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$$

$$\Rightarrow \cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

P1.

If A, B, C are angles of a triangle, then prove that

$$\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$$

Solution:

We have, $A + B + C = \pi$

$sin\ 2A + sin\ 2B - sin\ 2C$

$$= 2sin(A + B)cos(A - B) - 2\ sinC\ cosC$$

$$A + B = \pi - C \Rightarrow \sin(A + B) = \sin(\pi - C) = sinC$$

$$= 2sinC\ cos(A - B) - 2\ sinC\ cosC$$

$$= 2sinC\ \{cos(A - B) - cosC\}$$

$$A + B = \pi - C \Rightarrow \cos(A + B) = \cos(\pi - C) = -cosC$$

$$= 2sinC\ \{cos(A - B) + \cos\ (A + B)\}$$

$$= 2\ sinC\ (2\ cosA\ cosB)$$

$$= 4\ cosA\ cosB\ sin\ C$$

P2.

If A, B, C are angles in a triangle, then prove that

$$sinA + sinB + sinC = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution:

We have, $A + B + C = \pi$

$sinA + sinB + sinC$

$$= 2\, sin\left(\frac{A+B}{2}\right)\, cos\left(\frac{A-B}{2}\right) + \; 2\, sin\frac{C}{2}\, cos\frac{C}{2}$$

$$= 2\, cos\frac{C}{2}\, cos\left(\frac{A-B}{2}\right) + \; 2\, sin\frac{C}{2}\, cos\frac{C}{2}$$

(since $sin\frac{A+B}{2} = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = cos\frac{C}{2}$)

$$= 2\, cos\frac{C}{2}\left\{cos\left(\frac{A-B}{2}\right) + sin\frac{C}{2}\right\}$$

$$= 2\, cos\frac{C}{2}\left\{cos\left(\frac{A-B}{2}\right) + cos\left(\frac{A+B}{2}\right)\right\}$$

(since $cos\frac{A+B}{2} = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = sin\frac{C}{2}$)

$$= 2\, cos\frac{C}{2} . \left(2\cos\frac{A}{2}\,\cos\frac{B}{2}\right)$$

$$= 4\cos\frac{A}{2}\,\cos\frac{B}{2}\, cos\frac{C}{2}$$

P3.

In a triangle ABC, then prove that

$$cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}=4\ cos\frac{\pi-A}{4}\ cos\frac{\pi-B}{4}\ cos\frac{\pi-C}{4}$$

Solution:

We have, $A + B + C = \pi$

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

$$= \cos\frac{A}{2} + 2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$A + B + C = \pi \Rightarrow \cos\frac{A}{2} = \cos\left(\frac{\pi}{2} - \frac{B+C}{2}\right) = \sin\frac{B+C}{2}$$

$$= \sin\frac{B+C}{2} + 2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$= 2\sin\frac{B+C}{4}\cos\frac{B+C}{4} + 2\cos\frac{B+C}{2}\cos\frac{B-C}{4}$$

$$= 2\cos\frac{B+C}{4}\left(\sin\frac{B+C}{4} + \cos\frac{B-C}{4}\right)$$

$$= 2\cos\frac{B+C}{4} + \left[\cos\left(\frac{\pi}{2} - \frac{B+C}{4}\right) + \cos\frac{B-C}{4}\right]$$

$$= 2\cos\frac{B+C}{4}\, 2\cos\frac{\pi-C}{4}\cos\frac{\pi-B}{4}$$

$$= 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$

P4.

If $A + B + C = \pi$, then prove that

$$sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2} = 1 - 2\ \sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}.$$

Solution:

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$$

$$=\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\frac{1-\cos C}{2}$$

$$=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)$$

$$=\frac{3}{2}-\frac{1}{2}\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) \qquad \text{(As in IP2)}$$

$$=\frac{3}{2}-\frac{1}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Aliter:

We have given, $A+B+C=\pi$

$$\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2} \Longrightarrow \cos\frac{A+B}{2}=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\frac{C}{2}$$

$$\Longrightarrow \cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}=\sin\frac{C}{2}$$

$$\Longrightarrow \cos^2\frac{A}{2}\cos^2\frac{B}{2}=\left(\sin\frac{C}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right)^2$$

$$\Longrightarrow \cos^2\frac{A}{2}\cos^2\frac{B}{2}=\sin^2\frac{C}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Longrightarrow \left(1-\sin^2\frac{A}{2}\right)\left(1-\sin^2\frac{B}{2}\right)$$
$$=\sin^2\frac{C}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Longrightarrow 1-\sin^2\frac{A}{2}-\sin^2\frac{B}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}$$
$$=\sin^2\frac{C}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Longrightarrow 1-\sin^2\frac{A}{2}-\sin^2\frac{B}{2}=\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Longrightarrow \sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

IP1.

If A, B, C are angles of a triangle, then prove that

$$\cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C - 1$$

Solution:

We have, $A + B + C = \pi$

$$\cos 2A + \cos 2B + \cos 2C$$

$$= 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$$

$$= -2\cos C \cos(A-B) + 2\cos^2 C - 1$$

(since $\cos(A+B) = \cos(\pi - C) = -\cos C$)

$$= -2\cos C\ \{\cos(A-B) - \cos C\} - 1$$

$$= -2\cos C\ \{\cos(A-B) + \cos(A+B)\} - 1$$

$$= -2\cos C\ (2\cos A \cos B) - 1$$

$$= -4\cos A \cos B \cos C - 1.$$

IP2.

If A, B, C are angles in a triangle, then prove that

$$cosA + cosB + cosC = 1 + 4\, sin\frac{A}{2}\ sin\frac{B}{2}\ sin\frac{C}{2}$$

Solution:

We have, $A + B + C = \pi$

$cosA + cosB + cosC$

$$= 2\cos\left(\frac{A+B}{2}\right)\ cos\left(\frac{A-B}{2}\right) + 1 - \ 2\, sin^2\frac{C}{2}$$

(since $A + B + C = \pi \Longrightarrow \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2} \Longrightarrow cos\frac{A+B}{2} = sin\frac{C}{2}$

$$= 1 + 2\, sin\frac{C}{2}\ cos\left(\frac{A-B}{2}\right) - 2\, sin^2\frac{C}{2}$$

$$= 1 + 2\, sin\frac{C}{2}\ \left\{cos\left(\frac{A-B}{2}\right) - sin\frac{C}{2}\right\}$$

$$= 1 + 2\, sin\frac{C}{2}\ \left\{cos\left(\frac{A-B}{2}\right) - cos\left(\frac{A+B}{2}\right)\right\}$$

$$= 1 + 2\, sin\frac{C}{2}\left(2\sin\frac{A}{2}\ \sin\frac{B}{2}\right)$$

$$= 1 + 4\ \sin\frac{A}{2}\sin\frac{B}{2}\ sin\frac{C}{2}.$$

IP3.

If $A + B + C = \pi$, then prove that

$$\sum cos\frac{A}{2}\ cos\frac{B-C}{2} = 4\ cos\frac{A}{2}\ cos\frac{B}{2}\ cos\frac{C}{2}$$

Solution:

$$\sum cos\frac{A}{2}\ cos\frac{B-C}{2} = \frac{1}{2}\sum 2\ sin\frac{B+C}{2}\ cos\frac{B-C}{2}$$

$$= \frac{1}{2}\sum (sinB + sinC)$$

$$= \frac{1}{2}(sinB + sinC + sinC + sinA + sinA + sinB)$$

$$= sinA + sinB + sinC$$

$$= 4\ cos\frac{A}{2}\ cos\frac{B}{2}\ cos\frac{C}{2} \quad \text{(As in P2)}$$

IP4.

If $A+B+C=\pi$, then prove that

$$sin^2\frac{A}{2}+sin^2\frac{B}{2}-sin^2\frac{C}{2}=1-2cos\frac{A}{2}\ cos\frac{B}{2}\sin\frac{C}{2}$$

Solution:

Given, $A+B+C=\pi$

$$A+B=\pi-C\Longrightarrow \cos\left(\frac{A}{2}+\frac{B}{2}\right)=cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=sin\frac{C}{2}$$

$$\Longrightarrow cos\frac{A}{2}cos\frac{B}{2}-sin\frac{A}{2}sin\frac{B}{2}=sin\frac{C}{2}$$

$$\Longrightarrow cos\frac{A}{2}cos\frac{B}{2}-sin\frac{C}{2}=sin\frac{A}{2}sin\frac{B}{2}$$

$$\Longrightarrow \left(cos\frac{A}{2}cos\frac{B}{2}-sin\frac{C}{2}\right)^2=sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Longrightarrow cos^2\frac{A}{2}cos^2\frac{B}{2}+sin^2\frac{C}{2}-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}=sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Longrightarrow \left(1-sin^2\frac{A}{2}\right)\left(1-sin^2\frac{B}{2}\right)+sin^2\frac{C}{2}-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$$
$$=sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Longrightarrow 1-sin^2\frac{A}{2}-sin^2\frac{B}{2}+sin^2\frac{A}{2}\ sin^2\frac{B}{2}+sin^2\frac{C}{2}-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$$
$$=sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Longrightarrow sin^2\frac{A}{2}+sin^2\frac{B}{2}-sin^2\frac{C}{2}=1-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$$

Exercises

If $+B+C=180^{\circ}$, prove that

1. $cos^2A+cos^2B+cos^2C=1-2\cos A\cos B\cos C$.
2. $cos^2A+cos^2B-cos^2C=1-2\sin A\sin B\cos C$.
3. $\tan\frac{A}{2}\ \tan\frac{B}{2}+\tan\frac{B}{2}\ \tan\frac{C}{2}+\tan\frac{C}{2}\ \tan\frac{A}{2}=1$.
4. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\ \cot\frac{A}{2}\ \cot\frac{B}{2}\ \cot\frac{C}{2}$.
5. $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$

   $=4\ \sin\frac{B-C}{2}\ \sin\frac{C-A}{2}\ \cos\frac{A-B}{2}$
6. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1=4\ \sin\frac{\pi-A}{4}\ \sin\frac{\pi-B}{4}\ \sin\frac{\pi-C}{4}$.
7. $\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\ \cos\frac{\pi+A}{4}\ \cos\frac{\pi+B}{4}\ \cos\frac{\pi-C}{4}$.
8. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\cos C}=8\ \sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}$.
9. $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$

   $=4\sin A\sin B\sin C$.
10. In a triangle ABC if $\cot A+\cot B+\cot C=\sqrt{3}$, prove that the triangle is equilateral.

6.3
Trigonometric Equations

Learning objectives:

1. To find principle solution and general solution of a trigonometric equation.
2. To use different methods to solve trigonometric equations.
   And
3. To practice the related problems.

Solving Trigonometric Equations

Trigonometric equations are equations involving trigonometric functions of unknown angles and they, unlike identities, are satisfied only by particular values of the unknown angles. For example, $\sin x \,.\, \csc x = 1$ is an identity, being satisfied by every value of x for which $\sin x$ and $\csc x$ are defined.

$\sin x = 0$; it is not satisfied by $x = \frac{\pi}{4}$ or $\frac{\pi}{2}$. Since it is not satisfied by every value of x for which it is defined, it is not an identity. It is an equation and we will find the particular values of x for which this equation is satisfied.

A ***solution*** of a trigonometric equation is a value of the angle x which satisfies the equation.

If a given equation has one solution, it has in general an unlimited number of solutions due to the periodicity of the trigonometric functions.

Two solutions of $\sin x = 0$ are $x = 0$ and $x = \pi$.

The complete solution of $\sin x = 0$ is given by

$$x = 0 + 2n\pi, \qquad x = \pi + 2n\pi$$

where n is any integer. Both these expressions can be combined into a single expression $x = n\pi$, where n is any integer.

The solution consisting of all possible solutions of a trigonometric equation is called its ***general solution***.

There is no general method for solving trigonometric equations. Several standard procedures are employed in the solution of trigonometric equations.

The numerically least angle of the solution is called the ***principal value or principle solution***.

For example, find the principal value of $\sin x = \frac{1}{2}$.

The numerically least value will be in the first quadrant. Therefore, the principal value is $x = \frac{\pi}{6}$

- Find the principal value of x satisfying $\sin x = -\frac{1}{2}$.

The sine is negative in 3rd or 4th quadrant. Therefore, the principal value is $x = -\frac{\pi}{6}$

- Find the principal value of x satisfying $\tan x = -1$.

Tan is negative in 2^{nd} and 4^{th} quadrants. The principal value is $x = -\frac{\pi}{4}$

- Find the principal value of x satisfying $\cos x = \frac{1}{2}$.

Cosine is positive in first and fourth quadrants. The principal value is $x = \frac{\pi}{3}$

Principal value lies in the first quadrant. It is never numerically greater than π. The clockwise or anticlockwise direction is chosen depending on whether the angle is in 3rd and 4th quadrant or in first and second quadrants.

Factorable Equations

Solve $\sin x - 2\sin x\, cos\ x = 0.$

Factoring,

$$\sin x - 2\sin x \cos x = sinx(1 - 2cosx) = 0$$

Setting each factor equal to zero, we get

$$\sin x = 0 \Rightarrow x = 0, \pi$$

$$1 - 2\cos x = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Expressible in terms of a Single Function

Solve $2\, tan^2 x + sec^2 x = 2$

$2\, tan^2 x + sec^2 x = 2$

$2\, tan^2 x + 1 + tan^2 x = 2 \Rightarrow\ 3tan^2 x = 1$

$\Rightarrow\ \tan x = \pm\frac{1}{\sqrt{3}}$

For $\tan x = \frac{1}{\sqrt{3}}$, $x = \frac{\pi}{6}, \frac{7\pi}{6}$

For $\tan x = -\frac{1}{\sqrt{3}}$, $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

Solve $\sec x + \tan x = 0$

$$\frac{1}{cos\ x} + \frac{\sin x}{cos\ x} = 0$$

Multiplying by $\cos x$, we have $1 + \sin x = 0,\ \sin x = -1.$

Then $x = \frac{3\pi}{2}$

However, neither $\sec x$ nor $\tan x$ is defined when $x = \frac{3\pi}{2}$ and the equation has no solution.

This illustrates that there is a need to check the solution before accepting it as a solution of the equation.

Squaring Both Members of the Equation

Solve $\sin x + \cos x = 1$

We write the equation in the form $\sin x = 1 - \cos x$ and square both members. We have

$$sin^2 x = 1 - 2\cos x + cos^2 x$$

$\Rightarrow 1 - cos^2 x = 1 - 2\cos x + cos^2 x$

$\Rightarrow 2\, cos^2 x - 2\cos x = 0 \Rightarrow\ 2\cos x(\cos x - 1) = 0$

From $2cos\ x = 0$, $x = \frac{\pi}{2}, \frac{3\pi}{2}$

From $\cos x = 1$, $x = 0$

Check:

For $x = 0$, $\sin x + cosx = 0 + 1 = 1$

For $x = \frac{\pi}{2}$, $\sin x + cosx = 1 + 0 = 1$

For $x = \frac{3\pi}{2}$, $\sin x + cosx = -1 + 0 \neq 1$

Thus, the solution is $x = 0$ and $\frac{\pi}{2}$.

The value $x = \frac{3\pi}{2}$, called an *extraneous solution*, was introduced by squaring the members.

General solution of the equations

$\boldsymbol{sin\ x = 0, cos\ x = 0}$ and $\boldsymbol{tan\ x = 0}$

(i). If $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then $sin\theta = 0$ if and only if $\theta = 0$. Thus the principal solution of $sin\ x = 0$ is 0. Let $\theta \in R$ be any solution of $sin\ x = 0$. Then these exists a $k \in Z$ such that

$$k \leq \frac{\theta}{2\pi} < k + 1$$

$$\Rightarrow 2\pi k \leq \theta < 2\pi k + 2\pi$$

$$i.e., 0 \leq \theta - 2k\pi < 2\pi$$

Since θ and $\theta - 2\pi$ are co terminal angles,

$$\sin\theta = \sin(\theta - 2k\pi) = 0$$

$\Rightarrow\ \theta - 2k\pi = 0$ or $\pi \Rightarrow \theta = 2k\pi$ or $(2k+1)\pi, k \in Z$

i.e., $\theta = n\pi, n \in Z$

Thus, $sin\theta = 0 \ \Leftrightarrow \theta = n\pi, n \in Z$

Therefore, the general solution of the equation

$$\boldsymbol{sinx = 0\ is\ x = n\pi, n \in Z}$$

(ii). The principal solution of $cosx = 0$ is $x = \frac{\pi}{2}$. Further,

$$cos\ x = 0 \ \Leftrightarrow \sin\left(x - \frac{\pi}{2}\right) = 0 \Leftrightarrow x - \frac{\pi}{2} = n\pi,\ n \in Z$$

$$\Leftrightarrow x = n\pi + \frac{\pi}{2}$$

$$\Leftrightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$$

Thus, the general solution of $cosx = 0$ is

$$\boldsymbol{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

(iii). The principal value of $tan\ x = 0$ is $x = 0$. Further,

$$tanx = 0 \Leftrightarrow sinx = 0 \ \Leftrightarrow x = n\pi, n \in Z$$

Thus, the general solution of $tanx = 0$ is

$$\boldsymbol{x = n\pi, \qquad n \in Z}$$

NOTE: The general solution of $cot\ x = 0$ is given by

$$\boldsymbol{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

General solution of $sinx = k, |k| \leq 1$

Since $|k| \leq 1$, there exists a principal solution say α,

i.e., There is a $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $sin\,\alpha = k$

Let θ be any solution of $sin\,x = k$, then

$$sin\theta = sin\alpha \Leftrightarrow sin\theta - sin\alpha = 0$$

$$\Leftrightarrow 2cos\frac{\theta+\alpha}{2} . sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow 2cos\frac{\theta+\alpha}{2} = 0 \text{ or } sin\frac{\theta-\alpha}{2} = 0$$

Now, $cos\frac{\theta+\alpha}{2} = 0 \Leftrightarrow \frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}, n \in Z$

$$\Leftrightarrow \theta = (2n+1)\pi - \alpha, n \in Z$$

and, $sin\frac{\theta-\alpha}{2} = 0 \Leftrightarrow \frac{\theta-\alpha}{2} = n\pi, n \in Z$

$$\Leftrightarrow \theta = 2n\pi + \alpha, n \in Z$$

Combining those two, $\theta = n\pi + (-1)^n\alpha, n \in Z$

Thus, the general solution of the equation $sinx = k$, $|k| \leq 1$ is

$$\boldsymbol{x = n\pi + (-1)^n\alpha}$$

Where α is the principal solution (or any solution) of the equation.

By similar argument we prove the following

- The general solution of the equation of $cosx = k, |k| \leq 1$ is $\boldsymbol{x = 2n\pi \pm \alpha, n \in Z}$

where α is the principal solution (or any solution) of the equation

By similar argument we prove the following

- The general solution of the equation of $cosx = k, |k| \leq 1$ is $\boldsymbol{x = 2n\pi \pm \alpha, n \in Z}$

where α is the principal solution (or any solution) of the equation

- The general solution of the equation $tanx = k, \quad k \in R$ is $\boldsymbol{x = n\pi + \alpha, \ n \in Z}$

where α is the principal solution (or any solution) of the equation.

Summary

The equation $f(x)=k$	Range of k	The interval in which the principal solution α lies	General solution
$\sin x = k$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$n\pi+(-1)^n\alpha\ ,n\epsilon Z$
$\cos x = k$	$[-1,1]$	$[0,\pi]$	$2n\pi \pm \alpha\ ,n\epsilon Z$
$\tan x = k$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$n\pi + \alpha\ ,n\epsilon Z$
$\csc x = k$	$(-\infty,-1] \cup [1,\infty)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	$n\pi+(-1)^n\alpha\ ,n\epsilon Z$
$\sec x = k$	$(-\infty,-1] \cup [1,\infty)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$	$2n\pi \pm \alpha\ ,n\epsilon Z$
$\cot x = k$	R	$(0,\pi)$	$n\pi + \alpha\ ,n\epsilon Z$

Example:

Solve the equation $\sin x + \sin 5x = \sin 3x$.

$$\Longrightarrow 2\sin 3x \cos 2x = \sin 3x$$

$$\Longrightarrow \sin 3x\,(2\cos 2x - 1) = 0$$

Therefore, $\sin 3x = 0$, or $\cos 2x = \frac{1}{2}$

If $\sin 3x = 0$, then $3x = n\pi, n\epsilon z$.

If $\cos 2x = \frac{1}{2}$, then $2x = 2n\pi \pm \frac{\pi}{3}\ ,n\epsilon\ z$

Hence $x = \frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{6}\ ,n\epsilon\ z$.

The general solution of $\boldsymbol{sin^2x = k, 0 \le k \le 1}$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$, where $\boldsymbol{\alpha}$ is a solution of $\boldsymbol{sin^2x = k}$

Proof:

The trigonometric equation $sinx = k$ has a solution if and only if $k \in [0,1]$. Thus there exists a solution say $\alpha \in R$ such that $sin^2x = sin^2\alpha$. Now

$$sin^2x = sin^2\alpha \Leftrightarrow 1 - 2sin^2x = 1 - 2sin^2\alpha$$

$$\Leftrightarrow cos2x = cos2\alpha.$$

$$\Leftrightarrow 2x = 2n\pi \pm 2\alpha,\ n \in Z$$

$$\Leftrightarrow \boldsymbol{x = n\pi \pm \alpha,\ n \in Z}$$

where α is a solution of $sin^2x = k$.

By a similar method we prove the following

- The general solution of $cos^2x = k, 0 \le k \le 1$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$, where α is a solution of $cos^2x = k$

- The general solution of $tan^2x = k, 0 \le k < \infty$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$, where α is a solution of $tan^2x = k$

Equations of the form $a\cos\theta + b\sin\theta = c$

We divide both sides of the equation by $\sqrt{a^2+b^2}$, so that it may be written as

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

If we introduce the angle α, so that $\tan\alpha = \frac{b}{a}$

Then $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ $\qquad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

Also, we introduce the angle β, so that

$$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$$

The equation can then be written

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \cos\beta$$

The equation is then $\cos(\theta-\alpha) = \cos\beta$.

The solution of this is $\theta - \alpha = 2n\pi \pm \beta$, so that

$$\boldsymbol{\theta = 2n\pi + \alpha \pm \beta}$$

where n is any integer.

Angles, such as α and β, which are introduced to facilitate computation are called **Subsidiary Angles**.

Example: Solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$

We have $\sqrt{a^2+b^2} = \sqrt{1+3} = 2$. We therefore write the equations as

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

Therefore

$$\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4} \Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

Taking the positive sign,

$$x = 2n\pi + \frac{\pi}{6} + \frac{\pi}{4} = 2n\pi + \frac{5\pi}{12}$$

Taking the negative sign,

$$x = 2n\pi + \frac{\pi}{6} - \frac{\pi}{4} = 2n\pi - \frac{\pi}{12}$$

P1.

Find the general solution of $2cos^2x\,tanx = tanx$.

Solution:

Given, $2cos^2x\ tanx = tanx$

$\Longrightarrow tanx(2cos^2x - 1) = 0 \Longrightarrow tanx = 0\ or\ (2cos^2x - 1) = 0$

$\Longrightarrow tanx = 0\ or\ cosx = \frac{1}{\sqrt{2}} \Longrightarrow x = 0, \pi\ or\ x = \frac{\pi}{4}$

$\Longrightarrow x = n\pi\ or\ x = n\pi \pm \frac{\pi}{4}, n \in Z$

P2.

Solve $3\cos 2\theta + 2 = 7\sin\theta$

Solution:

$$3\cos 2\theta + 2 = 7\ \sin\theta$$

$$\Longrightarrow 3\,(1 - 2sin^2\theta) + 2 = 7\,sin\theta$$

$$\Longrightarrow\ 6\,sin^2\theta + 7\,sin\theta - 5 = 0$$

$$\Longrightarrow (2\sin\theta - 1)(3\sin\theta + 5) = 0$$

$\Longrightarrow\ \sin\theta = \frac{1}{2}, -\frac{5}{3}$ and $\sin\theta = -\frac{5}{3} \notin [-1, 1]$

$\therefore\ \sin\theta = \frac{1}{2} \Longrightarrow\ \theta = \frac{\pi}{6}$ (Principal solution)

The general solution is $n\pi + (-1)^n\ \frac{\pi}{6}\ , n \in Z$

P3.

Solve: $\sqrt{3}\cos x - sinx = 1$

Solution:

Given, $\sqrt{3}\cos x - sinx = 1$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}sinx = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}sinx = \cos\frac{\pi}{3}$$

$$\Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}$$

P4.

If θ_1, θ_2 are solutions of the equation

$acos\theta + bsin\theta = c, tan\theta_1 \neq tan\theta_2$ and $a + c \neq 0,$ then find

the values of

i. $tan\theta_1 + tan\theta_2$

ii. $tan\theta_1 \cdot tan\theta_2$

Solution:

Given, $acos\theta + bsin\theta = c\,, a + c \neq 0$

$$\Rightarrow a\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right) + b\left(\frac{2tan\theta}{1+tan^2\theta}\right) = c$$

$$\Rightarrow a - atan^2\theta + 2btan\theta = c + ctan^2\theta$$

$$\Rightarrow (a+c)tan^2\theta - 2btan\theta + c - a = 0$$

$$\Rightarrow (a+c)tan^2\theta - 2btan\theta + c - a = 0$$

This is a quadratic equation in $tan\,\theta$. Since θ_1, θ_2 are roots of the given equation, we get $tan\theta_1$ and $tan\theta_2$ are roots of the equation.

$\therefore$ sum of the roots $tan\theta_1 + tan\theta_2 = \frac{2b}{a+c}$ and

Product of the roots $tan\theta_1 \cdot tan\theta_2 = \frac{c-a}{a+c}$

IP1.

Find the general solution of $2cos^2u = 1 - cosu$.

Solution:

Given, $2cos^2u = 1 - cosu$

$\Longrightarrow 2cos^2u + cosu - 1 = 0$

$\Longrightarrow (2cosu - 1)(cosu + 1) = 0$

$\Longrightarrow 2cosu - 1 = 0$ or $cosu + 1 = 0$

$\Longrightarrow cosu = \frac{1}{2}$ or $cosu = -1$

$\Longrightarrow u = \frac{\pi}{3}$ or $u = \pi$

General solution is $\left\{2n\pi \pm \frac{\pi}{3} \mid n \in Z\right\} \cup \{2n\pi \pm \pi \mid n \in Z\}$

IP2.

Solve: $2cos^2\theta + 11\ sin\theta = 7$

Solution:

$$2cos^2\theta + 11\ sin\theta = 7$$

$$\Rightarrow 2\,(1 - sin^2\theta) + 11\ \ sin\theta - 7 = 0$$

$$\Rightarrow\ 2 - 2\,sin^2\theta + 11\ sin\theta - 7 = 0$$

$$\Rightarrow 2\ sin^2\theta - 11\ sin\theta + 5 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta - 5) = 0$$

$$\Rightarrow 2\sin\theta - 1 = 0\ \ or\ \ sin\theta - 5 = 0$$

$\Rightarrow\ \sin\theta = \frac{1}{2}$ or $sin\theta = 5 > 1$

$$\Rightarrow\ \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}\ \ (Principle\ solutions)$$

The general solution is, $\theta = n\pi + (-1)^n\ \frac{\pi}{6}\ ,\ n \in Z$

IP3.

Solve: $2cosx + 2sinx = \sqrt{6}$

Solution:

Given, $2cosx + 2sinx = \sqrt{6}$

$$\Rightarrow \frac{cosx}{\sqrt{2}} + \frac{sinx}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos\frac{\pi}{4}cosx + sin\frac{\pi}{4}sinx = cos\frac{\pi}{6}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = cos\frac{\pi}{6}$$

$$\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6}$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{6}$$

$$\therefore\ x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{6}$$

IP4.

If $\alpha\ and\ \beta$ are the solutions of $atan\theta + bsec\theta = c$, then show that $\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$

Solution:

Given, $a\ tan\theta + b\ sec\theta = c$

$$bsec\theta = c - atan\theta$$

$$\Longrightarrow b^2sec^2\theta = c^2 + a^2tan^2\theta - 2actan\theta$$

$$\Longrightarrow b^2(1 + tan^2\theta) = c^2 + a^2tan^2\theta - 2ac\ tan\theta$$

$$\Longrightarrow (a^2 - b^2)tan^2\theta - 2ac\ tan\theta + (c^2 - b^2) = 0$$

Also given, α and β are the solutions of θ.

$tan\alpha + tan\beta = \frac{2ac}{a^2-c^2}$ and $tan\alpha \cdot tan\beta = \frac{c^2-b^2}{a^2-b^2}$

$$\tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\ \cdot tan\beta} = \frac{\left(\frac{2ac}{a^2-c^2}\right)}{1 - \left(\frac{c^2-b^2}{a^2-b^2}\right)} = \frac{2ac}{a^2-c^2}$$

1. Solve the following equations.

a) $sin\theta + sin7\theta = sin\ 4\theta$

b) $cos\theta + cos7\theta = cos4\theta$

c) $cos\theta + cos3\theta = 2\cos 2\theta$

d) $\sin 4\theta - \sin 2\theta = \cos 3\theta$

e) $cos\theta - sin3\theta = \cos 2\theta$

f) $\sin 7\theta = \sin\theta + \sin 3\theta$

g) $\cos\theta + \cos 3\theta = 0$

h) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

i) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$

j) $\cos n\theta = \cos(n-2)\theta + \sin\theta$

k) $\sin\frac{n+1}{2}\theta = \sin\frac{n-1}{2}\theta + \sin\theta$

l) $\sin m\theta + \sin n\theta = 0$

m) $\cos m\theta + \cos n\theta = 0$

n) $\sin 3\theta + \cos 2\theta = 0$

o) $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

p) $\sin\theta + \cos\theta = \sqrt{2}$

q) $sin^2\ n\theta - sin^2\ (n-1)\theta = sin^2\ \theta$

6.4

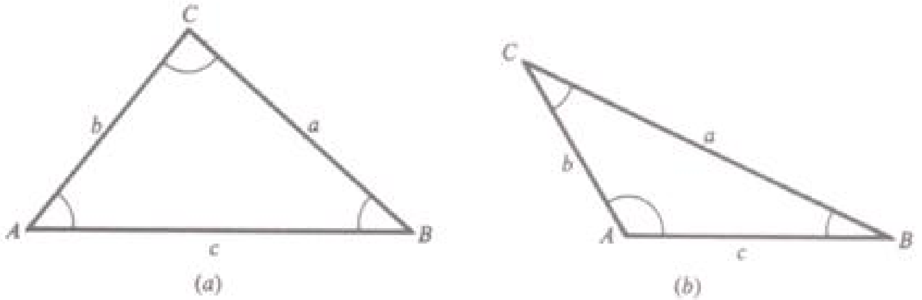
Relations between the Angles and Sides of a Triangle

Learning Objectives:

1. To derive law of sines, law of cosines and to find sines, cosines, tangents of half angles in terms of sides.
2. To derive tangent rules and projection formulas
   And
3. To practice the related problems.

Notation

A right triangle is the one which has a right angle as one of its angles. On the other hand, a general triangle is one which does not contain a right angle. Such a triangle contains either three acute angles or two acute angles and one obtuse angle.



It is a standard convention to denote the length of sides opposite to angles *A*, *B*, and *C* by *a*, *b*, and *c* respectively.

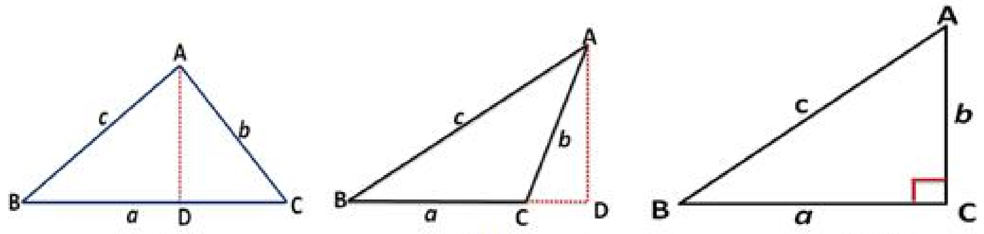
The Law of Sines

Theorem: In any triangle ABC

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

i.e., the sines of the angles are proportional to the opposite sides

Proof:



(i) (ii) (iii)

Draw AD perpendicular to the opposite side meeting it, produced if necessary in the point D.

In $\Delta\ ABD$, we have $\frac{AD}{AB} = \sin B \implies AD = c \sin B$

In $\Delta\ ACD$, we have $\frac{AD}{AC} = \sin C \implies AD = b \sin C$

If the angle C is obtuse, then as in the second figure, we have

$$\frac{AD}{AC} = \sin ACD = \sin(\pi - C) = \sin C$$

$$\implies AD = b \sin C$$

Thus $c \sin B = b \sin C$, i.e., $\frac{\sin B}{b} = \frac{\sin C}{c}$

Similarly, by drawing a perpendicular from B onto CA, we prove that $\frac{\sin C}{c} = \frac{\sin A}{a}$

If one of the angles, say C is a right angle as in the third figure then $\sin C = 1,\ \sin A = \frac{a}{c},\ \sin B = \frac{b}{c}$

Therefore, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c} = \frac{\sin C}{c}$

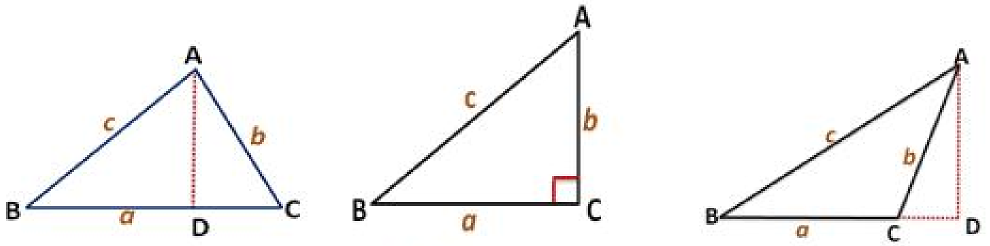
We have, in all cases

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Hence the theorem

The law of Cosines

In any triangle ABC, the square of any side is equal to the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of the included angle.



(i) (ii) (iii)

Let ABC be the triangle and let the perpendicular from A on BC meet it, produced if necessary, in the point *D.*

(*i*) Let the angle C be acute, as in fig (i) then

$$BD = BC - DC = a - b\ cosC$$

In the right triangle ABD, we have

$$\begin{aligned} c^2 &= BD^2 + AD^2 \\ &= (a - b\ cosC)^2 + b^2\ sin^2C \\ &= a^2 - 2\ ab\ cosC + b^2\ cos^2C + b^2\ sin^2C \\ &= a^2 + b^2 - 2\ ab\ cosC \end{aligned}$$

(*ii*) Let the angle C be obtuse as in fig (ii).

Then $\quad BD = BC + CD = a + b\ cos(\pi - C)$

$$= a - b\ cosC$$

In the right angle ABD,we have

$$\begin{aligned} c^2 &= BD^2 + AD^2 \\ &= (a - b\ cosC)^2 + b^2 sin^2C \\ &= a^2 + b^2 - 2\ ab\ cosC \quad \text{as in } (i) \end{aligned}$$

(*iii*) Let the angle C be a right angle as in fig (*iii*)

Then $\ cosC = 0$ and

$$c^2 = a^2 + b^2 = a^2 + b^2 - 2\ ab\ cosC$$

In any case,

$$\boldsymbol{c^2 = a^2 + b^2 - 2\ ab\ cosC}$$

Similarly it may be shown that

$$\boldsymbol{a^2 = b^2 + c^2 - 2\ bc\ cos\ A}$$

$$\boldsymbol{b^2 = c^2 + a^2 - 2\ ca\ cos\ B}$$

Example:

The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double the smallest angle.

Solution:

Let $a = 8, b = 10, c = 12$

Here the greatest angle is C and the smallest angle is A and we have to prove C = 2A. Now

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{64+100-144}{2\times8\times10} = \frac{1}{8}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{100+144-64}{2\times10\times12} = \frac{3}{4}$$

and $cos2A = 2cos^2A - 1 = 2\times\left(\frac{3}{4}\right)^2 - 1 = cosC$

$\Rightarrow 2A = C$

Sines, Cosines, Tangents of Half Angles

(i). $2sin^2\frac{A}{2} = 1 - cosA = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc-b^2-c^2+a^2}{2bc}$

$$= \frac{a^2-(b^2+c^2-2bc)}{2bc} = \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

Let $2s$ stand for $a+b+c$, so that $s = \frac{a+b+c}{2}$, Then

$a+b-c = 2(s-c)$ and $c+a-b = 2(s-b)$

Therefore,

$2sin^2\frac{A}{2} = \frac{2(s-c)\times2(s-b)}{2bc} \Rightarrow \boldsymbol{sin}\frac{\boldsymbol{A}}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Similarly, $\boldsymbol{sin}\frac{\boldsymbol{B}}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\boldsymbol{sin}\frac{\boldsymbol{C}}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii). $2cos^2\frac{A}{2} = 1 + cosA = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{2bc+b^2+c^2-a^2}{2bc}$

$$= \frac{(b+c)^2-a^2}{2bc} = \frac{(a+b+c)(b+c-a)}{2bc} = \frac{2s\times2(s-a)}{2bc}$$

$\Rightarrow \boldsymbol{cos}\frac{\boldsymbol{A}}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Similarly,

$\boldsymbol{cos}\frac{\boldsymbol{B}}{2} = \sqrt{\frac{s(s-b)}{ca}}$ and $\boldsymbol{cos}\frac{\boldsymbol{C}}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii). $\boldsymbol{tan}\frac{\boldsymbol{A}}{2} = \frac{sin\frac{A}{2}}{cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly,

$\boldsymbol{tan}\frac{\boldsymbol{B}}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ and $\boldsymbol{tan}\frac{\boldsymbol{C}}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Since, in a triangle, A is always less than 180°, A/2 is always less than 90°. The sine, cosine, and tangent of A/2 are therefore always positive.

Some Useful Identities

We will express the sine of any angle of a triangle in terms of its sides.

$$sinA = 2sin\frac{A}{2}cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \boldsymbol{sinA = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}}$$

The following identity is useful in solving the problems

$$tan\frac{B-C}{2} = \frac{b-c}{b+c}cot\frac{A}{2}$$

We prove this identity as follows.

In any triangle, we have $\frac{b}{c} = \frac{sin\ B}{sinC}$

Therefore,

$$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2cos\frac{B+C}{2}\ sin\frac{B-C}{2}}{2sin\frac{B+C}{2}\ cos\frac{B-C}{2}}$$

$$= \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{\tan\frac{B-C}{2}}{tan\left(90° - \frac{A}{2}\right)} = \frac{\tan\frac{B-C}{2}}{cot\frac{A}{2}}$$

Hence

$$\boldsymbol{tan\frac{B-C}{2} = \frac{b-c}{b+c}cot\frac{A}{2}}$$

This is known as ***Tangent Rule***. Two other formulas can be written in a similar manner.

Projection Formulae

In any triangle ABC, $\boldsymbol{a = bcosC + ccosB}$

$$\boldsymbol{b = ccosA + acosC}$$

$$\boldsymbol{c = acosB + bcosA}$$

Proof:

From cosine rules

$cosC = \frac{a^2+b^2-c^2}{2ab}$, $cosB = \frac{a^2+c^2-b^2}{2ac}$

$$\therefore b\ cosC + c\ cosB = \frac{a^2+b^2-c^2}{2a} + \frac{a^2+c^2-b^2}{2a}$$

$$= \frac{2a^2}{2a} = a$$

Similarly, the other laws can be proved.

P1.

Show that $(b-c)^2 \, cos^2 \frac{A}{2} + (b+c)^2 \, sin^2 \frac{A}{2} = a^2$

Solution:

$$(b^2+c^2-2bc)cos^2\frac{A}{2}+(b^2+c^2+2bc)sin^2\frac{A}{2}$$

$$=(b^2+c^2)\left(cos^2\frac{A}{2}+sin^2\frac{A}{2}\right)-2bc\left(cos^2\frac{A}{2}-sin^2\frac{A}{2}\right)$$

$$=b^2+c^2-2bc\ cosA=a^2$$

P2.

In $\Delta\ ABC$, find $bcos^2\frac{C}{2}+ccos^2\frac{B}{2}$

Solution:

$$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=b\left[\frac{s(s-c)}{ab}\right]+c\left[\frac{s(s-b)}{ca}\right]$$

$$=\frac{s(s-c)}{a}+\frac{s(s-b)}{a}$$

$$=\frac{s}{a}[s-c+s-b]$$

$$=\frac{s}{a}[2s-c-b]$$

$$=\frac{s}{a}[a+b+c-c-b]$$

$$=\frac{s}{a}.a=s$$

$$\therefore b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=s$$

P3.

Prove that $a\cos\frac{B-C}{2} = (b+c)\sin\frac{A}{2}$

Solution:

We have,

$$\frac{b+c}{a} = \frac{sinB + sinC}{sinA} = \frac{2\,sin\frac{B+C}{2}\ cos\frac{B-C}{2}}{2\,sin\frac{A}{2}\ cos\frac{A}{2}}$$

$$= \frac{2\,cos\frac{A}{2}\,cos\frac{B-C}{2}}{2\,sin\frac{A}{2}\,cos\frac{A}{2}}$$

$$= \frac{cos\frac{B-C}{2}}{sin\frac{A}{2}}$$

$$\therefore a\,cos\frac{B-C}{2} = (b+c)sin\frac{A}{2}$$

P4.

In $\Delta\, ABC$, show that $\sum(b+c)cosA = 2s$

Solution:

In $\Delta\, ABC$,

$$\sum (b+c)cosA = (b+c)cosA + (c+a)cosB + (a+b)cosC$$

L.H.S. = $(b+c)cosA + (c+a)cosB + (a+b)cosC$

$$= (bcosA + acosB) + (c\,cosB + bcosC) + (a\,cosC + c\,cosA)$$

$= c + a + b$ (From projection formulas)

$$= 2s$$

$$\therefore \sum (b+c)cosA = 2s$$

IP1.

If $a = 6, b = 5, c = 9$, then find angle A?

Solution:

Given, $a = 6, b = 5, c = 9$

From cosine rule,

$$cosA = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{25+81-36}{2.5.9} = \frac{70}{90} = \frac{7}{9}$$

$$\therefore A = Cos^{-1}\left(\frac{7}{9}\right)$$

IP2.

If $tan\frac{A}{2} = \frac{5}{6}$ and $tan\frac{C}{2} = \frac{2}{5}$, determine the relation among a, b, c.

Solution:

Given, $tan\frac{A}{2} = \frac{5}{6}$ and $tan\frac{C}{2} = \frac{2}{5}$

$$tan\frac{A}{2}.tan\frac{C}{2} = \frac{5}{6}.\frac{2}{5} = \frac{2}{6}$$

i.e., $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{2}{6}$

$$\Longrightarrow \frac{s-b}{s} = \frac{1}{3} \Longrightarrow 3s - 3b = s \Longrightarrow 2s = 3b$$

$$\Longrightarrow a + b + c = 3b \Longrightarrow a + c = 2b.$$

Hence a, b, c are in $A.P.$

IP3.

$$\frac{a+b}{a-b} = tan\frac{A+B}{2}cot\frac{A-B}{2}$$

Solution:

Given,

$$\frac{a+b}{a-b} = \frac{sinA + sinB}{sinA - sinB} = \frac{2\,sin\frac{A+B}{2}\ cos\frac{A-B}{2}}{2\,cos\frac{A+B}{2}\ sin\frac{A-B}{2}}$$

$$= \frac{sin\frac{A+B}{2}}{cos\frac{A+B}{2}} \cdot \frac{cos\frac{A-B}{2}}{sin\frac{A-B}{2}}$$

$$= tan\frac{A+B}{2} \cdot cot\frac{A-B}{2}$$

$$\therefore \quad \frac{a+b}{a-b} = tan\frac{A+B}{2}cot\frac{A-B}{2}$$

IP4.

In a $\Delta\, ABC$, prove that $\sum a^3 \cos(B-C) = 3abc$

Solution:

Given,

$$a^3 \cos(B-C) = a^2\, a\cos(B-C) = a^2 k\, sinA\cos(B-C)$$

where, $k = \frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

$$= a^2 k\, sin(B+C)\cos(B-C)$$

$$(since\ A+B+C=\pi)$$

$$= \frac{a^2 k}{2}(2sin(B+C)\cos(B-C))$$

$$= \frac{a^2 k}{2}(sin2B + sin2C)$$

$$= \frac{a^2 k}{2}(2sinB\cos B + 2sinC\, cosC)$$

$$= a^2(b\cos B + c\, cosC)$$

Similarly,

$b^3 \cos(C-A) = b^2(a\cos A + c\, cosC)$,

$c^3 \cos(A-B) = c^2(a\cos A + b\, cosB)$

$$\sum a^3 \cos(B-C) = a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$$

$$= a^2(b\cos B + c\, cosC) + b^2(a\cos A + c\, cosC)$$

$$+c^2(a\cos A + b\, cosB)$$

$$= a^2 b\cos B + a^2 c\, cosC + b^2 a\cos A + b^2 c\, cosC$$

$$+c^2 a\cos A + c^2 b\, cosB$$

$$= ab(a\, cosB + b\, cosA) + bc(b\, cosC + c\, cosB) + ca(c\, cosA + a\, cosC)$$

$$= abc + bca + cab = 3abc$$

$$\therefore \sum a^3 \cos(B-C) = 3abc$$

Exercises

1. In a triangle ABC, if, $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ prove that $\frac{cosA}{7}=\frac{cosB}{19}=\frac{cosC}{25}$

2. In any triangle ABC, if $tan\theta = \frac{2\sqrt{ab}}{a-b} sin\frac{C}{2}$, prove that $c=(a-b)sec\theta$

3. If $a=13, b=14, c=15$ find the trigonometric ratios of the half angles of the triangle.

4. Given $a=\sqrt{3}, b= \sqrt{2}, c=\frac{\sqrt{6}+\sqrt{2}}{2}$, find the angles.

5. In any triangle prove that
$$(b^2-c^2)cotA+(c^2-a^2)cotB+(a^2-b^2)cotC=0$$

6. In any triangle prove that $(a+b+c)\left(tan\frac{A}{2}+tan\frac{B}{2}\right)=2c\,cot\frac{C}{2}$

7. In any triangle ABC, prove that

a) $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$

b) $b^2\sin 2C + c^2\sin 2B = 2\,bc\sin A$

c) $a\,(b\cos C - c\cos B) = b^2 - c^2$

d) $a\,(\cos B + \cos C) = 2\,(b+c)\,sin^2\frac{A}{2}$

e) $\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$

f) $c^2 = (a-b)^2\,cos^2\frac{C}{2} + (a+b)^2\,sin^2\frac{C}{2}$

g) $\frac{a\sin(B-C)}{b^2-c^2} + \frac{b\sin(C-A)}{c^2-a^2} + \frac{c\sin(A-B)}{a^2-b^2}$

h) $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C = 0$

i) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$

j) If a, b and c are in H.P., prove that $sin^2\frac{A}{2}, sin^2\frac{B}{2}$ and $sin^2\frac{C}{2}$ are also in H.P.

6.5
Solution of Triangles

Learning objectives:

1. To find the measurements of the angles and sides of a triangle if any three measurements (not all angles) are given
   And
2. To practice the related problems.

In any triangle the three sides and the three angles are called the ***elements*** of the triangle. When any three elements, not all angles, of the triangle are given, the remaining elements can be calculated. Given three elements, the process of calculating the remaining three elements is called the ***solution of the triangle***.

I. Given Three Sides

The semi-perimeter is determined from the three known sides. The half-angles are then found from the half-angle formulas. We can also use the law of cosines.

Example:

The sides of a triangle are 32, 40, and 66 cm; find the angle opposite to the greatest side.

Solution:

Let $a = 32, b = 40, c = 66$

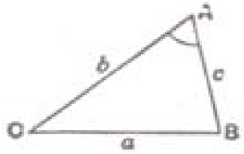
$$s = \frac{32+40+66}{2} = 69$$

$$s - a = 37, s - b = 29, s - c = 3$$

$$tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{37\times29}{69\times3}} = 2.2768$$

$$\frac{C}{2} = 66°17'19'' \Rightarrow C = 132°34'38''$$

II. Given Two Sides *b* and *c* and the Included Angle *A*



Taking *b* to be the greater side of the two given sides, we have, $tan\frac{B-C}{2}=\frac{b-c}{b+c}cot\frac{A}{2}$

This gives $\frac{B-C}{2}$, and this when solved simultaneously with

$$\frac{B+C}{2}=90°-\frac{A}{2}$$

yield the values of *B* and *C*. The third side *a* is then known from the relation $\frac{a}{sinA}=\frac{b}{sinB}$

The side *a* may also be found from the cosine formula.

Example:

If $b=\sqrt{3}, c=1, A=30°$, solve the triangle.

Solution:

$$tan\frac{B-C}{2}=\frac{b-c}{b+c}\,cot\frac{A}{2}=\frac{\sqrt{3}-1}{\sqrt{3}+1}cot15°$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}.\frac{\sqrt{3}+1}{\sqrt{3}-1}=1$$

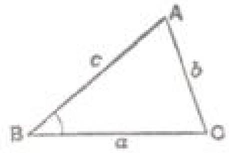
$$\frac{B-C}{2}=45°$$

$$\frac{B+C}{2}=90°-15°=75°$$

$$B=120°, C=30°$$

Since $C=A,$ we have $a=c=1.$

III. Given two sides *b* and *c* and the angle *B* opposite to one of them



From the relation $\frac{sinC}{c} = \frac{sinB}{b}$,

we obtain the relation $sinC = \frac{c}{b} sinB$ --- (1)

From which we can determine C.

Once the angle C is determined, *A* can be found from the relation $A = 180° - B - C$

The remaining side a is then found from the relation

$$\frac{a}{sinA} = \frac{b}{sin\ B}$$

It is possible to have no solution, one solution or two solutions (an angle and its supplement) for C.

Let *B* be an acute angle.

a) If $b < c\ sin\ B$, then $sinC > 1$ from (1) and hence there is no solution for C.

b) If $b = c \sin B$, then $sinC = 1$ from (1) and then $C\ = 90°$.

c) If $b > c\ SinB$, then $sinC < 1$ from (1) and there are two values of C, one value lying between $0°$ and $90°$ and the other between $90°$ and $180°$.

Both these values are not always admissible.

If $b > c$, then $B > C$. Therefore, the obtuse-angled value of C is now not admissible.

If $b < c$, then $C > B$, both values of C are admissible. In this case, there are two triangles satisfying the given conditions.

Let *B* be an obtuse angle

If $b \leq c$, then $B \leq C$ and C would be an obtuse angle. Since no two obtuse angles are permissible in a triangle, there is no solution.

If $b > c$, the acute value of C (determined from (1)) would be admissible, but not the obtuse value. Therefore there is only one admissible solution.

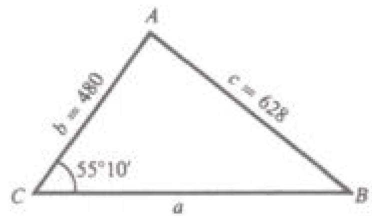
Example:

Solve the triangle ABC, given

$c = 628, b = 480 \text{ and } C = 55°10'$

Solution:

Since C is acute and $c > b$, there is only one solution.



$$sinB = \frac{bsinC}{c} = \frac{480sin55°10'}{628} = \frac{480\times0.8208}{628} = 0.6274$$

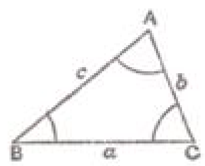
$$B = 38°50'$$

$$A = 180° - (B + C) = 86°0'$$

$$a = \frac{bsin\ A}{sinB} = \frac{480\ \sin 86°0'}{sin38°50'} = \frac{480\times0.9976}{0.6271} = 764$$

IV. Given one side and two angles

Let a, B, C are given



The third angle is determined from $A = 180° - (B + C)$

The sides b and c are now obtained from the relations

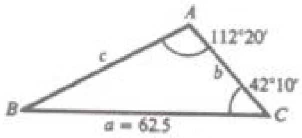
$$\frac{b}{sinB} = \frac{c}{sinC} = \frac{a}{sinA}$$

Example:

Solve the triangle ABC, given

$$a = 62.5, A = 112°20', C = 42°10'$$

Solution:



$$B = 180° - (C + A) = 180° - 154°30' = 25°30'$$

$$b = \frac{a\sin B}{\sin A} = \frac{62.5 \times sin25°30'}{sin112°20'} = \frac{62.5 \times 0.4305}{0.9250} = 29.1$$

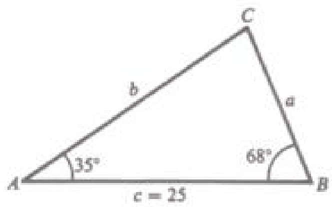
$$c = \frac{a\ sinC}{\sin A} = \frac{62.5 \times sin42°10'}{\sin 112°20'} = \frac{62.5 \times 0.6713}{0.9250} = 45.4$$

Example:

Solve the triangle ABC, given

$c = 25, A = 35°, B = 68°$

Solution:



$$C = 180° - (B + A) = 180° - 103° = 77°$$

$$a = \frac{c \sin A}{\sin C} = \frac{25 \times sin35°}{sin77°} = \frac{25 \times 0.5736}{0.9744} = 15$$

$$b = \frac{c\ sin\ B}{\sin C} = \frac{25 \times sin68°}{sin77°} = \frac{25 \times 0.9272}{0.9744} = 24$$

V. The three angles are given

In this case the ratios of the sides can be determined by the formulae

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$$

Their absolute magnitudes can't be found.

P1.

Find the greatest angle of a triangle whose sides are 242,188 and 270.

Solution:

Given,

$a = 242, b = 188$ and $c = 270$.

The greatest angle of the triangle is the angle opposite to the largest side.

∴ The greatest angle is C.

From given data $s = \frac{a+b+c}{2} = \frac{242+188+270}{2} = 350$

$$tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{108\times162}{350\times80}} = \sqrt{0.62485} = 0.79047$$

$\frac{C}{2} = 38°19'31'' \Rightarrow C = 76°39'2''$

Therefore, the greatest angle of the triangle $C = 76°39'2''$

P2)

If $a = 13, b = 7$ and $C = 60°. Find$ A and B

Solution:

Given, $a = 13, b = 7 \; and \; C = 60°$

we have, $Tan \frac{A-B}{2} = \frac{a-b}{a+b} cot \frac{C}{2}$

$\Rightarrow Tan \frac{A-B}{2} = \frac{13-7}{20} cot 30^0$

$\Rightarrow Tan \frac{A-B}{2} = \frac{6}{20} \times \sqrt{3}$

$\Rightarrow Tan \frac{A-B}{2} = \frac{3\sqrt{3}}{10}$

$\Rightarrow Tan \frac{A-B}{2} = 0.51961 \Rightarrow \frac{A-B}{2} = 27°27'24''$

$\Rightarrow A - B = 54°54'48''$ ... (1)

Now, $A + B = 180° - C = 120°$ ... (2)

add (1) & (2),

$2A = 174°54'48'' \Rightarrow A = 87°27'24''$

$B = 120^0 - A = 120° - 87°27'24'' = 32°32'36''$

P3)

If $a = 2, c = \sqrt{3} + 1$ and $A = 45°$ then solve the triangle.

Solution:

Given, $a = 2, c = \sqrt{3} + 1$ and $A = 45°$

By sine rule we have,

$\frac{sinA}{a} = \frac{sinC}{c} \Rightarrow sinC = \frac{c}{a}\ sinA = \frac{\sqrt{3}+1}{2} sin45^0 = \frac{\sqrt{3}+1}{2} \times \frac{1}{\sqrt{2}}$

$\sin C = \frac{\sqrt{3}+1}{2\sqrt{2}} \Rightarrow C = 75°$ and $C = 105°$

Now $a < c$ and A is acute, both the values of C are admissible.

Case (i): $C = 75°$

Now, $B = 180° – (A + C) = 180° – (120°) = 60°$

By sine rule,

$\frac{sinB}{b} = \frac{sinA}{a} \Rightarrow b = a.\frac{sinB}{sinA}$

$$= 2 \times \frac{\sin 60^0}{\sin 45^0} = 2 \times \frac{\sqrt{3}}{2} \times \sqrt{2} = \sqrt{6}$$

Case (ii): $C = 105°$

Now, $B = 180° – (A + C) = 180° – (150°) = 30°$

By sine rule,

$\frac{sinB}{b} = \frac{sinA}{a} \Rightarrow b = a.\frac{sinB}{sinA}$

$$= 2 \times \frac{\sin 30^0}{\sin 45^0} = 2 \times \frac{1}{2} \times \sqrt{2} = \sqrt{2}$$

P4.

In $\Delta\, EFG,\, e = 4.56, E = 43°$ and $G = 57°$ then find the value of side f ?

Solution:

Given,

$e = 4.56, E = 43°$ and $G = 57°$

From $\Delta\, EFG$

$$F = 180° - (E + G) = 180° - 100° = 80°$$

Using sin law $\frac{f}{\sin F} = \frac{e}{\sin E} \Longrightarrow f = e\,\frac{\sin F}{\sin E}$

$$\Longrightarrow f = 4.56 \times \frac{\sin 80°}{\sin 43°}$$

$$= 4.56 \times \frac{0.9848}{0.6819} \approx 6.58$$

IP1.

Find all the angles when $a = 17, b = 20\ and\ c = 27.$

Solution:

Given, $a = 17, b = 20\ and\ c = 27$

Now, $s = \frac{a+b+c}{2} = \frac{17+20+27}{2} = \frac{64}{2} = 32$

$$Tan\ \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{\frac{12\times 5}{32\times 15}} = \sqrt{\frac{1}{8}} = \sqrt{0.125} = 0.3535$$

$$\Rightarrow \frac{A}{2} = 19°28'6''$$

$$\Rightarrow A = 38°56'12''$$

$$Tan\ \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= \sqrt{\frac{15\times 5}{32\times 12}} = \sqrt{\frac{25}{128}}$$

$$= \sqrt{0.1953125} = 0.4419$$

$$\Rightarrow \frac{B}{2} = 23°50'26'' \ \Rightarrow B = 47°40'52''$$

$$C = 180° - (A + B)$$

$$= 180° - (86°37'04'') = 93°22'56''$$

IP2)

If $a = 2b$ and $C = 120°$ then find the values of A, B?

Solution:

Let, $a = 2b = k$

Then, $a = k, b = \frac{k}{2}, C = 120°$

we have, $tan\ \frac{A-B}{2} = \frac{a-b}{a+b} cot\frac{C}{2} = \frac{k-\frac{k}{2}}{\frac{3k}{2}} cot60^0 = \frac{\frac{k}{2}}{\frac{3k}{2}} cot60^0$

$\Rightarrow Tan\ \frac{A-B}{2} = \frac{1}{3} cot60^0$

$\Rightarrow Tan\ \frac{A-B}{2} = \frac{1}{3} . \frac{1}{\sqrt{3}}$

$\Rightarrow Tan\ \frac{A-B}{2} = \frac{1}{3\sqrt{3}} = 0.19245 \Rightarrow \frac{A-B}{2} = 10°53'36''$

$\Rightarrow A - B = 21°47'12''$ ... (1)

Now, $A + B = 180° - 120° = 60°$ ... (2)

Add (1) and (2),

$2A = 81°47'12'' \Rightarrow A = 40°53'36''$

$B = 60^0 - 40°53'36'' = 19°06'24''$

IP3

If $a = 9, b = 12\ and\ A = 30°\ find\ c$

Solution:

Given, $a = 9, b = 12\ and\ A = 30°$

By sine rule we have, $\frac{sinA}{a} = \frac{sinB}{b}$

$\therefore \sin B = \frac{b}{a} sinA = \frac{12}{9} sin30^0 = \frac{12}{9} \times \frac{1}{2} = \frac{2}{3}$

$\Rightarrow sinB = 0.6666$

$\Rightarrow B = 41°48'18''$ and $B = 138°11'42''$

Now $a < b$ and A is acute, both the values of B are admissible.

Case (i): $B = 41°48'18''$

Now, $C = 180^0 - (A + B)$

$= 180^0 - (71°48'18'') = 108°11'42''$

By sine rule, $c = a.\frac{sinC}{sinA} = 9 \times \frac{0.9494}{0.5} = 17.08$

Case (ii): $B = 138°11'42''$

Now, $C = 180^0 - (A + B)$

$= 180^0 - (168°11'42'') = 11°48'18''$

By sine rule, $c = a.\frac{sinC}{sinA} = 9 \times \frac{0.2053}{0.5} = 3.6963$

IP4.

If the angles of a triangle ABC are in the ratio $1:2:7$, then the ratio of the greatest side to the least side is

Solution:

Let A denote the angle in the triangle. Then by the hypothesis, we have $A + 2A + 7A = 180° \Rightarrow A = 18°$

$\therefore$ The angles of the triangle are $A = 18°,\ B = 36°,$

$C = 126°$ and the a is the least and c is the greatest side.

Hence $\frac{a}{sinA} = \frac{c}{sinC} \Rightarrow \frac{a}{sin18°} = \frac{c}{sin126°}$

$$\Rightarrow \frac{c}{a} = \frac{sin126°}{sin18°} = \frac{cos36°}{sin18°} = \frac{\left(\frac{\sqrt{5}+1}{4}\right)}{\left(\frac{\sqrt{5}-1}{4}\right)} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

$$\Rightarrow c : a = (\sqrt{5}+1):(\sqrt{5}-1)$$

Exercise

I. Making the use of the values to find greatest and smallest and all the angles of triangle.

1. $a = 25, b = 26$ and $c = 27$
2. $a = 2000, b = 1050$ and $c = 1150$
3. $a = 7, b = 4\sqrt{3}$ and $c = \sqrt{13}$
4. $a = 2, b = \sqrt{6}$ and $c = \sqrt{3} - 1$

II. Find all the angles from given values of triangle

1. $b = 14, c = 11$ and $A = 60°$ then find B and C.
2. $a = 2526, c = 1388$ and $B = 54.42°$ then find A and C.
3. $a = 525, c = 421, A = 130°50'$ then find B and C.
4. $\mathrm{a} = 31.5, b = 51.8, A = 33°40'$ then find B and C.

III. Solve the following triangles.

1. $a = 5, b = 4$ and $A = 45°$
2. $a = 9, b = 12$ and $A = 30°$
3. $a = 100, c = 100\sqrt{2}$ and $A = 30°$
4. $b = 8, a = 6$ and $A = 30°$

IV. Solve the following triangles.

1. $a = 38.1, A = 46.5°, C = 74.3°$.

2. $b = 282.7, A = 111.72°, C = 24.43°$.